Math 2050, summary of Week 7

1. Series

Definition 1.1. *Given a sequence* $\{x_n\}_{n=1}^{\infty}$*, the series generated by* $\{x_n\}_{n=1}^{\infty}$ *is given by* $s_i = \sum_{k=1}^{i} x_k$.

Examples: The followings are the most important (and fundamental) examples of series.

(1) Geometric series: $\sum_{i=1}^{k} r^i$;
(2) Harmonic series $\sum_{n=1}^{k} n^{-1}$;
(3) p-series $\sum_{n=1}^{k} n^{-p}$;

Clearly, series is a special case of sequence. We are interested in their convergence since they are special and appear quite often.

We start with the elementary nature of series.

Theorem 1.1 (The n-the term test)**.** *Suppose* $\sum x_n$ *converges, then* $x_n \to 0$.

E.g. $\sum (-1)^n$ is divergent since $(-1)^n$ does not converge to 0. But it is far from equivalent. For example, $\sum k^{-1}$ is unbounded and divergent (as shown in previous lecture), but $k^{-1} \to 0$ as $k \to +\infty$.

Since the theory of sequence is better developed (at this stage), we now translate the corresponding Theorem in the setting of series.

Theorem 1.2 (Cauchy criterion)**.** *The series* $\sum x_n$ *is convergent if and only if* $\forall \varepsilon > 0, \exists N$ *such that for all* $m > n > N$*, we have*

$$|\sum_{k=n+1}^{m} x_k| < \varepsilon.$$

Theorem 1.3 (monotone convergence theorem)**.** *Suppose* $x_n \geq 0$ *for all* $n \in \mathbb{N}$*, then the series* $\sum x_n$ *is convergent if and only if there is* $M > 0$ *such that for all* $m \in \mathbb{N}$*,*

$$\sum_{k=1}^{m} x_k \leq M.$$

Example (Useful trick): $\sum k^{-2}$ is convergent. By MCT, it suffices to show the boundedness.

$$\begin{aligned}\sum_{k=1}^{m}\frac{1}{k^2} &\le 1+\sum_{k=2}^{m}\frac{1}{k^2}\\ &\le 1+\sum_{k=2}^{m}\frac{1}{k(k-1)}\\ &\le 1+\sum_{k=2}^{m}\left(\frac{1}{k-1}-\frac{1}{k}\right)\\ &\le 2-\frac{1}{m}\\ &< 2.\end{aligned} \tag{1.1}$$

And hence it is convergent.

This can be generalized further as one can see the argument only relies on some comparison after some large index.

Theorem 1.4 (Comparison Test)**.** *Suppose $\{x_n\}$ and $\{y_n\}$ are sequence such $0 \le x_n \le y_n$ for all $n > k_0$. Then we have*

(1) *$\sum x_n$ is convergent if $\sum y_n$ is convergent.*
(2) *$\sum y_n$ is divergent if $\sum x_n$ is divergent.*

Therefore, one only need to find some "reference" series to determine the convergence.

The convergence of the fundamental Examples:

(a) $\sum r^k$ is convergent if $r < 1$ and is divergent if $r \ge 1$;
(b) $\sum k^{-p}$ is convergent if $p > 1$ and is divergent if $p \le 1$.

2. Function

Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function on A.

Ultimate Objective: Study the continuity of f.

We are only interested in some "meaningful" point.

Definition 2.1. *Let $A \subset \mathbb{R}$. A real number c is said to be a cluster point of A if for all $\delta > 0$, there is $x \in A$ such that $0 < |x - c| < \delta$.*

It is easy to see that equivalently we can approximate c by sequence in $A \setminus \{c\}$.

Theorem 2.1. *Let $A \subset \mathbb{R}$. Then c is cluster point of A if and only if there is $\{a_n\} \subset A$ such that $a_n \neq c$ and $a_n \to c$ as $n \to +\infty$.*

Examples:

(1) $A = (0,1)$, then cluster points are $[0,1]$;
(2) $A = \{p_i\}_{i=1}^k$, then there are no cluster points;
(3) $A = \{k^{-1} : k \in \mathbb{N}\}$, then cluster point is $\{0\}$;
(4) $A = (0,1) \cap \mathbb{Q}$, then cluster points are $[0,1]$.

Cluster points are those points which is NOT isolated. Those are what we care!

Theorem 2.2. *Let $A \subset \mathbb{R}$ and c is a cluster point of A, $f : A \to \mathbb{R}$. Then $L \in \mathbb{R}$ is said to be the limit of f at c if for all $\varepsilon > 0$, there is $\delta > 0$, such that for all $x \in A$ with $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$. In this case, we will denote $\lim_{x \to c} f = L$.*

The notion is reasonable since the limit is unique if exists.

Theorem 2.3. *Let $A \subset \mathbb{R}$ and c is a cluster point of A, $f : A \to \mathbb{R}$. Then f can at most have a single limit at c.*

The definition is usually not user friendly when we try to argue the opposite. We therefore need some alternative perspective of the definition.

Theorem 2.4 (Sequence criterion)**.** *Let $A \subset \mathbb{R}$ and c is a cluster point of A, $f : A \to \mathbb{R}$. Then we have $\lim_{x \to c} f = L$ if and only if for any sequence $\{a_n\} \subset A \setminus \{c\}$ so that $a_n \to c$, we have $f(a_n) \to L$.*

The contra-positive statement is given as follows.

Theorem 2.5 (Divergent criterion)**.** *Let $A \subset \mathbb{R}$ and c is a cluster point of A, $f : A \to \mathbb{R}$. Then*

(1) *f does not have a limit L at c if and only if $\exists \varepsilon_0 > 0$, $\{x_n\} \subset A \setminus \{c\}$ such that $x_n \to c$ but $|f(x_n) - L| \geq \varepsilon_0$ for all n.*
(2) *f does not have a limit at c if and only if $\exists \{x_n\} \subset A \setminus \{c\}$ such that $x_n \to c$ but $\{f(x_n)\}_{n=1}^{\infty}$ is divergent.*

Examples: *Direct application of the divergent criterion is to show that $\lim_{x \to 0} x^{-1}$ and $\lim_{x \to 0} \sin(x^{-1})$ does not exist.*